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ON STRONG MAPS OF MATROIDS

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Abstract

The aim of this paper is to introduce the notions of A-strong maps and B-strong maps between matroids as the maps that have the preimage of every flat is an A-flat (B-flat, respectively). Several significant results for A- and B-strong maps are provided. Moreover, we provide new decompositions of strong maps via these notions.

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1 Introduction

The matroid terminology and notation will follow Oxley [7]. For an introduction on matroids, see also [6, 7, 8, 9]. In particular, a *matroid* M is an ordered pair $(E,\mathfrak{F})$, denoted by $M(E,\mathfrak{F}_M)$, where $\mathfrak{F}_M$ is a collection of subsets, called *flats* of M, of a finite set E such that $E \in \mathfrak{F}_M$, intersections of flats are flats and if $F \in \mathfrak{F}_M$ and $\{F_1, F_2, ..., F_k\}$

is the set of minimal members of $\mathfrak{F}_M$ (with respect to inclusion) that properly contain F, then $F_1 \cup F_2 \cup ... \cup F_k = E$. A subset $A \subseteq E$ is open if and only if $E \backslash A$ is a flat.

Let $M(E,\mathfrak{F})$ be a matroid. The *closure* of a subset $A \subseteq E$ will be denoted by $\bar{A}$. Clearly $\bar{A}$ is the smallest flat containing A, see Oxley [7]. M is called *a loopless matroid* if $\overline{\varnothing} = \varnothing$, i.e. $\varnothing$ is a flat. A is called an *LF-set* if $A = O \cap F$, where O is open and F is a flat, see [3].

Let $M_1 = M(E_1,\mathfrak{F}_1)$ and $M_2 = M(E_2,\mathfrak{F}_2)$ be matroids. A *strong map* f from M_1 to M_2 is a map $f : E_1 \to E_2$ such that the inverse image of each flat of M_2 is a flat of M_1. We abbreviate this as $f : M_1 \to M_2$.

Several decompositions of strong maps between matroids have been obtained:

Theorem 1 *For a map $f : M_1 \to M_2$ the following conditions are equivalent:*

(1) f is strong.

(2) f is w-strong and w^*-strong [5].

(3) f is NO-strong and ORF-strong [5], provided that M_1 is loopless.

(4) f is WOFF-strong and prestrong [1].

(5) f is OFR-strong and prestrong [2], provided that M_1 is loopless and every OFR-set is FO-set.

(6) f is OFR-strong and IF-strong [2], provided that M_1 is loopless and every OFR-set is FO-set.

(7) f is alternative strong and LF-strong [3], provided that M_1 is loopless.

(8) f is prestrong and LF-strong [3], provided that M_1 is loopless.

(9) f is prestrong and ORF-strong [3], provided that M_1 is loopless.

(10) $f(\overline{A}) \subseteq \overline{f(A)}$, for every subset $A \subseteq E_1$ [10].

Variations of strong maps of matroids are defined as analogues of variations of continuous maps of topological spaces. Thus, it is natural to ask what other topological notions carry over to matroids. The aim of this paper is to introduce two new variations of strong maps that are weaker than strong map notion and then provide new decompositions of strong maps via these weak forms, namely A- and B-strong maps.

2 A- and B-flats

In this section, we introduce the notions of A-flats and B-flats and discuss their connections to several known others in [1, 2, 3, 4, 5, 10]. Moreover, we give many characterizations of these notions.

Definition 1 *Let $M(E,\mathfrak{F})$ be a matroid. A subset A of E is called* A-flat *if $A = F \cap C$, where F is a flat and $C \subseteq E$ such that $C = \cap\{O|O$ is open and $C \subseteq O\}$. $E\backslash A$ is then called* A-open.

Lemma 1 *Let $M(E,\mathfrak{F})$ be a matroid. Then the following are equivalent for a subset $A \subseteq E$:*

(1) $A = \overline{A} \cap (\cap\{O|O$ is open and $A \subseteq O\})$.

(2) $A = \overline{A} \cap C$, where $C \subseteq E$ such that $C = \cap\{O|O$ is open and $C \subseteq O\}$

(3) A is an A-flat.

Proof. (1)$\Rightarrow$(2): Follows immediately by letting $C = \cap\{O|O$ *is open and* $A \subseteq O\}$.

(2)$\Rightarrow$(3): Obvious as $\overline{A}$ is a flat.

(3)$\Rightarrow$(1): Clearly $A \subseteq \cap\{O|O$ *is open and* $A \subseteq O\}$ and $A \subseteq \overline{A}$. Thus $A \subseteq \overline{A}\cap(\cap\{O|O$ *is open and* $A \subseteq O\})$. On the other hand, as $A = F \cap C$, where F is a flat and $C \subseteq E$ such that $C = \cap\{O|O$ *is open and* $C \subseteq O\}$ is a subset of F, we have

$$\begin{aligned}\overline{A} \cap (\cap\{O|O \text{ is open and } A &\subseteq O\}) \subseteq F \cap (\cap\{O|O \text{ is open and } A \subseteq O\}) \\ &\subseteq F \cap (\cap\{O|O \text{ is open and} C \subseteq O\}) \\ &= C \cap F \\ &= A.\end{aligned}$$

Therefore, $A = \overline{A} \cap (\cap\{O|O$ *is open and* $A \subseteq O\})$. □

Lemma 2 *In a matroid $M(E,\mathfrak{F})$,*

(1) If $C = \cap\{O|O$ is open and $C \subseteq O\}$, then C is an A-flat.

(2) Every LF-set is an A-flat.

(3) Every flat is an A-flat, provided $M(E,\mathfrak{F})$ is loopless.

(4) Every open set is an A-flat.

Proof. (1): Let $C = \cap\{O|O \text{ is open and } C \subseteq O\}$. Then $C = C \cap \overline{C}$ is an A-flat according to Lemma 1.

(2): Let A be an LF-set, then $A = O \cap F$ where O is open and F is a flat. Hence as $O = \cap\{U|U \text{ is open and } O \subseteq U\}$, A is an A-flat.

(3): Since in a loopless matroid every flat is an LF-set, by (2), it is an A-flat.

(4): Since an open set $O = O \cap E$, where E is a flat, A is an LF-set and then by (2) O is an A-flat. □

Next, we show that none of the converses of the parts in the preceding lemma need be true.

Example 1 *Let $E = \{a, b, c\}$ and $\mathfrak{F} = \{\varnothing, E, \{b, c\}, \{a, c\}, \{c\}\}$ and consider the matroid $M(E,\mathfrak{F})$. Set $C = \{c\}$. Then as C is an LF-set, by Lemma 2 (2), C is an A-flat, but $C \neq \cap\{O|O \text{ is open and } C \subseteq O\}$.*

Example 2 *Let $E = \{a, b, c\}$ and $\mathfrak{F} = \{\varnothing, E, \{a\}, \{b\}, \{c\}, \{d\}\}$ and consider the matroid $M(E,\mathfrak{F})$. Set $A = \{a, b\}$. Then A is an A-flat but not an LF-set.*

Next we characterize B-flats in terms of the closure operator.

Lemma 3 *Let $M(E,\mathfrak{F})$ be a matroid and $A \subseteq E$. Then A is a B-flat if and only if $\overline{A} \subseteq \cap\{O|O \text{ is open and } A \subseteq O\}$.*

Proof. If A is a B-flat and O is an open set such that $A \subseteq O$, then $A = A \cap E \backslash O = \varnothing$ where $E \backslash O$ is a flat. Thus $\overline{A} \cap E \backslash O = \varnothing$ and hence $\overline{A} \subseteq O$. Therefore, $\overline{A} \subseteq \cap\{O|O \text{ is open and } A \subseteq O\}$.

Conversely, if $\overline{A} \subseteq \cap\{O|O \text{ is open and } A \subseteq O\}$ and F is a flat such that $A \cap F = \varnothing$, then $A \subseteq E \backslash F$ which is open. Hence $\overline{A} \subseteq E \backslash F$ and so $\overline{A} \cap F = \varnothing$. □

By the aid of Lemma 3, we can now characterize the flats of a loopless matroid in terms of both A-flats and B-flats.

Theorem 2 *Let $M(E,\mathfrak{F})$ be a loopless matroid. Then the following are equivalent for $A \subseteq E$:*

(1) A is a flat.

(2) A is a B-flat and an LF-set.

(3) A is an A-flat and a B-flat.

Proof. (1)⇒(2): If A is a flat, then $\overline{A} = A$. If F is a flat such that $A \cap F = \varnothing$, then $\overline{A} \cap F = \varnothing$ and hence A is a B-flat. On the other hand as M is loopless, E is open and $A = E \cap \overline{A}$. Thus A is an LF-set.

(2)⇒(3): This part follows from Lemma 2 (2).

(3)⇒(1): Let A is an A-flat and a B-flat. By Lemma 3, $\overline{A} \subseteq \cap\{O | O \ \textit{is open and} \ A \subseteq O\}$ and by Lemma 1 $A = \overline{A} \cap (\cap\{O | O \ \textit{is open and} \ A \subseteq O\})$. Thus $A = \overline{A}$ and hence A is a flat. □

3 Strong map decompositions

In view of the preceding results, more interesting concepts than LF-, OFR-,W-strong maps and many others characterizing essentially the same behavior of maps are given next.

Definition 2 *A map $f : M_1 = M(E_1,\mathfrak{F}_1) \to M_2 = M(E_2,\mathfrak{F}_2)$ is*

(1) B-strong *if the inverse of every flat in M_2 is a B-flat in M_1.*

(2) LF-strong *if the inverse of every open set in M_2 is an LF-set in M_1 [3].*

(3) CLF-strong *if the inverse of every flat in M_2 is an LF-set in M_1.*

(4) A-strong *if the inverse of every flat in M_2 is an A-flat in M_1.*

Since by Lemma 2, every LF-set is an A-flat, every CLF-strong map is A-strong, but the converse need not be true.

Example 3 *Consider the matroid $M(E,\mathfrak{F})$ from Example 2, let $\mathfrak{F}_2 = \{\varnothing, E, \{a\}, \{b, c, d\}\}$ and consider the map $f : M \to M(E, \mathfrak{F}_2)$ defined by $f(a) = f(b) = a$ and $f(c) = f(d) = c$. Then f is A-strong but not CLF-strong and not LF-strong.*

Notice that the map f in Example 3 which is A-strong is not B-strong as $f^{-1}(\{a\}) = \{a, b\}$ is not a B-flat.

Example 4 *Let $E = \{a, b, c\}$ and $\mathfrak{F}_1 = \{E, \{b, c\}, \{a, c\}, \{c\}\}$ and $\mathfrak{F}_2 = \{E, \{a\}\}$ and consider the identity map $f : M(E, \mathfrak{F}_1) \to M(E, \mathfrak{F}_2)$. Then f is B-strong but not A-strong.*

Clearly, every flat is a B-flat and hence every strong map is B-strong, but the converse need not be true.

Example 5 *Let $E = \{a, b\}$ and $\mathfrak{F}_1 = \{E, \varnothing\}$ and $\mathfrak{F}_2 = \{E, \{c\}\}$ and consider the identity map $f : M(E, \mathfrak{F}_1) \to M(E, \mathfrak{F}_2)$. Then f is B-strong but not strong.*

Our main result is the following decomposition of strong maps which follows immediately from definitions and Theorem 2.

Theorem 3 *Let M_1 be a loopless matroid. Then the following are equivalent for a map $f : M_1 \to M_2$:*

(1) f is strong.

(2) f is B-strong and CLF-strong.

(3) f is B-strong and A-strong.

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