On the Maximal and Efficient Sets in the Decision-Making Systems

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Abstract: The paper presents the maximal set and three efficient sets in the decision-making systems with finitely many criteria. Here, we study some properties of these optimality sets - maximal and efficient. In the end, it is proved that the efficient set is nonempty, path-connected and compact, if the utility functions are continuous, concave and strictly quasi-concave.

Keywords: maximal set, efficient set, binary relation, utility function, connected.

1. Introduction

Let a decision-making system (A,C) be given. In this system, we have that:

- The set A is a set of alternatives, $|A| > 1$.
- The set C is a finite set of criteria, $|C| = n \geq 2$.

Let $\{ R^k \}_{k=1}^{n}$ be a profile of preference of binary relation on A (from set A to itself) and R^k a relation to a criterion $c_k \in C$ representing “more preferred than” or “dominates” with respect to a criterion $c_k \in C$. Therefore, each criterion $c_k \in C$ has the relation R^k such that for every two alternatives $x, y \in A$ there is $xR^i y$ if and only if an alternative x dominates an alternative y to a criterion $c_k \in C$ (preferences x by y), see [1] and [4].

Let any relation R^k be reflective (if $x \in A$, then $xR^k x$), transitive (if $x, y, z \in A$, $xR^i y$ and $yR^i z$, then $xR^i z$) and complete (if $x, y \in A$, then $xR^i y$ holds or $yR^i x$ holds).

Let denote the asymmetric part of relation R^k by relation P^k (for $x, y \in A$ there is $xP^k y$ if and only if $xR^k y$ holds and $yR^k x$ does not hold). The relation P^k to a criterion $c_k \in C$ is transitive (if $x, y, z \in A$, $xP^k y$ and $yP^k z$, then $xP^k z$).

Let denote the symmetric part of relation R^k by relation I^k (for $x, y \in A$ there is $xI^k y$ if and only if $xR^k y$ and $yR^k x$ hold). The relation I^k to a criterion $c_k \in C$ is reflective (if $x \in A$, then $xI^k x$) and transitive (if $x, y, z \in A$, $xI^k y$ and $yI^k z$, then $xI^k z$).

For each $c_k \in C$ and $x \in A$ let also have that:

- The set $R_k(x)=\{y\in A: yR^k x\}$ is called a set of weakly preference to a criterion $c_k\in C$ (it is also called the upper contour set). The sets $R_k(x)$ and $\bigcap_{k=1}^n R_k(x)$ are nonempty subsets of A, $x\in R_k(x)$, $x\in\bigcap_{k=1}^n R_k(x)$.
- The set $P_k(x)=\{y\in A: yP^k x\}$ is called a set of strictly preference to a criterion $c_k\in C$. The sets $P_k(x)$ and $\bigcap_{k=1}^n P_k(x)$ can be empty, $x\notin P_k(x)$, $x\notin\bigcap_{k=1}^n P_k(x)$.
- The set $I_k(x)=\{y\in A: yI^k x\}$ is called a set of indifference to a criterion $c_k\in C$. The sets $I_k(x)$ and $\bigcap_{k=1}^n I_k(x)$ are nonempty subsets of A, $x\in I_k(x)$, $x\in\bigcap_{k=1}^n I_k(x)$.
- The set $L_k(x)=\{y\in A: xR^k y\}$ is called the lower contour set. The sets $L_k(x)$ and $\bigcup_{k=1}^n L_k(x)$ are nonempty subsets of A, $x\in L_k(x)$, $x\in\bigcup_{k=1}^n L_k(x)$.

It is easy to see that $I_k(x)\subset R_k(x)$, $P_k(x)\subset R_k(x)$, $I_k(x)\subset L_k(x)$, $R_k(x)\bigcap L_k(x)=I_k(x)$, $R_k(x)=I_k(x)\bigcup P_k(x)$, and $I_k(x)\bigcap P_k(x)$ is empty.

2. For maximal and efficient sets

An alternative $x\in A$ is called a maximal alternative in A if and only if there is $xR^k y$ for all $y\in A$ and all $c_k\in C$. The set of the maximal alternatives in A we shall denote by M. The set M is also called a maximal set.

It is easy to see that:

- If $x\in A$, then $x\in M \Leftrightarrow x\in\bigcap_{k=1}^n R_k(y)$ for all $y\in A$.

An alternative $x\in A$ weakly dominates an alternative $y\in A$ if and only if $xR^k y$ for all $c_k\in C$ and $x\neq y$. We call that the alternative $x\in A$ is weakly efficient if and only if there does not exist alternative $y\in A$ such that y weakly dominates x. The set of weakly efficient alternatives of A we shall denote by E_w.

It is easy to see that:

- If $x,y\in A$, then y weakly dominates $x \Leftrightarrow y\in\bigcap_{k=1}^n R_k(x)\setminus\{x\}$.
- If $x\in A$, then $x\in E_w \Leftrightarrow \{x\}=\bigcap_{k=1}^n R_k(x)$.
- If $x\in A$, then $x\in E_w \Leftrightarrow \left|\bigcap_{k=1}^n R_k(x)\right|=1$.

An alternative $x\in A$ dominates an alternative $y\in A$ if and only if $xR^k y$ for all $c_k\in C$ and $xP^j y$ for some $c_j\in C$. We call that the alternative $x\in A$ is efficient if and only if there does not exist alternative $y\in A$ such that y dominates x. The set of efficient alternatives of A we shall denote by E.

This is a standard definition of the efficient criterion [7] [12] [17].

It is easy to see that:

- If $x,y\in A$, then y dominates $x \Leftrightarrow y\in\bigcap_{k=1}^n R_k(x)\bigcap\bigcup_{k=1}^n P_k(x)$.
- If $x\in A$, then $x\in E \Leftrightarrow \bigcap_{k=1}^n R_k(x)=\bigcap_{k=1}^n I_k(x)$.
- $M\subset E$.

An alternative $x \in A$ strictly dominates an alternative $y \in A$ if and only if $xP^k y$ for all $c_k \in C$. We call that the alternative $x \in A$ is strictly efficient if and only if there does not exist alternative $y \in A$ such that y strictly dominates x. The set of strictly efficient alternatives of A we shall denote by E_s.

It is easy to see that:

- If $x, y \in A$, then y strictly dominates x $\Leftrightarrow$ $y \in \bigcap_{k=1}^{n} P_k(x)$.
- If $x \in A$, then $x \in E_s$ $\Leftrightarrow$ the set $\bigcap_{k=1}^{n} P_k(x)$ is empty.
- If $x \in A$, then $x \in E_s$ $\Leftrightarrow$ $\bigcap_{k=1}^{n} R_k(x) \subset \bigcup_{k=1}^{n} I_k(x)$.
- If $x \in A$, then $x \in E_s$ $\Leftrightarrow$ $\bigcup_{k=1}^{n} L_k(x) = A$.

These problems are also considered in [10]. The maximal set and efficient sets are also called optimality sets.

3. Some properties of the optimality sets

On the basic of definitions it is easy to show that:

- The set M can be empty or nonempty.
- If M is nonempty, then $M = E$.

Theorem 1 [10, Theorem 6]. (a) If $x \in E_w$ and $y \in \bigcap_{k=1}^{n} R_k(x)$, then $x = y$.

(b) If $x \in E$ and $y \in \bigcap_{k=1}^{n} R_k(x)$, then $y \in E$ and $y \in \bigcap_{k=1}^{n} I_k(x)$.

(c) If $x \in E_s$ and $y \in \bigcap_{k=1}^{n} R_k(x)$, then $y \in E_s$ and $y \in \bigcup_{k=1}^{n} I_k(x)$.

Analogously, we also obtain the following statements:

- If $x \in M$ and $y \in \bigcap_{k=1}^{n} R_k(x)$, then $y \in M$ and $y \in \bigcap_{k=1}^{n} I_k(x)$.
- If $x \in M$, then $\bigcap_{k=1}^{n} R_k(x) = \bigcap_{k=1}^{n} I_k(x)$.
- If $x, y \in M$, then $\bigcap_{k=1}^{n} R_k(x) = \bigcap_{k=1}^{n} I_k(x) = \bigcap_{k=1}^{n} R_k(y) = \bigcap_{k=1}^{n} I_k(y)$.

Theorem 2 [10, Theorem 8]. $E_w \subset E \subset E_s$.

Now let assume that $A \subset R^m$, $m \geq 1$, and A be convex.

A relation R^k is quasi-concave on A if and only if the set $R_k(x)$ is convex for all $x \in A$ ($y \in R_k(x)$ and $t \in [0;1]$ implies $tx + (1-t)y \in R_k(x)$). A relation R^k is strictly quasi-concave on A if and only if the set $R_k(x)$ is strictly convex for all $x \in A$ ($y \in R_k(x)$, $x \neq y$ and $t \in (0;1)$ implies $tx + (1-t)y \in P_k(x)$).

Theorem 3. (a) If the relations $\{R^k\}_{k=1}^{n}$ are quasi-concave on A and there exists a unique relation R^λ of $\{R^k\}_{k=1}^{n}$ which is strictly quasi-concave on A, then $E_w = E$.

(b) If the relations $\{R^k\}_{k=1}^{n}$ are strictly quasi-concave on A, then $E_w = E_s$.

Proof. (a) From Theorem 2 it is known that $E_w \subset E$.

Conversely, let $x \in E$ and assume that $x \notin E_w$. Therefore, there exists $y \in A$ such that $y \in \bigcap_{k=1}^{n} R_k(x)$ and $x \neq y$. Let $t \in (0;1)$ and $z = tx + (1-t)y$. From condition the relations $\{R^k\}_{k=1}^{n}$ are quasi-concave it follows that $z \in \bigcap_{k=1}^{n} R_k(x)$. From condition the relation R^λ is strictly quasi-concave it follows that $z \in P_k(x)$. As a result, we obtain $z \notin \bigcap_{k=1}^{n} I_k(x)$. This contradicts condition $\bigcap_{k=1}^{n} R_k(x) = \bigcap_{k=1}^{n} I_k(x)$. Hence, we have that $E \subset E_w$.

Finally, we obtain $E_w = E$.

(b) From Theorem 2 it is known that $E_w \subset E_s$.

Conversely, let $x \in E_s$ and assume that $x \notin E_w$. Therefore, there exists $y \in A$ such that $y \in \bigcap_{k=1}^{n} R_k(x)$ and $x \neq y$. Let $t \in (0;1)$ and $z = tx + (1-t)y$. From condition the relations $\{R^k\}_{k=1}^{n}$ are strictly quasi-concave it follows that $z \in \bigcap_{k=1}^{n} P_k(x)$. This contradicts condition $x \in E_s$. Hence, we have that $E_s \subset E_w$.

Finally, we obtain $E_w = E_s$. The theorem is proved.

From this theorem it is clear that if the relations $\{R^k\}_{k=1}^{n}$ are strictly quasi-concave on A, then $E_w = E = E_s$.

4. For utility functions

In this section, let assume that $A \subset R^m$, $m \geq 1$.

Let d be a metric in R^m and τ be a usual topology for R^m, see [6].

- In a topological space (R^m, τ), a point $x \in R^m$ is called a limit point of $X \subset R^m$ if and only if for each $T \in \tau$ such that $x \in T$ and $T \bigcap X$ is nonempty. The union of X and the set of all its limit points is called its closure and denote by clX. A point $x \in X$ is called an interior point of X if and only if there exists $T \in \tau$ such that $x \in T \subset X$. We denote the set of the interior points of X by inX. A point $x \in R^m$ is called a boundary point of X if and only if for each $T \in \tau$ such that $x \in T$, $T \bigcap X$ and $T \bigcap (R^m \setminus X)$ are nonempty. We denote the set of the boundary points of X by boX.
- In a topological space (R^m, τ), a set X is called a connected if and only if it is not the union of a pair of nonempty sets of τ which are disjoint, a set X is called a path-connected if and only if for any $x, y \in X$ there exists a continuous function $f : [0;1] \to X$ such that $f(0) = x$ and $f(1) = y$.

It is known that every relation R^k is reflective, transitive and complete therefore a set A is totally preordered by R^k.

Let also assume that the set A be connected.

Remark 1. If the set A is convex, then it is path-connected. If the set A is path-connected, then it is connected [6].

A relation R^k is closed if and only if the sets $\{y \in A : xR^k y\}$ and $\{y \in A : yR^k x\}$ are closed for all $x \in A$.

Let also assume that the binary relation $\{R^k\}_{k=1}^{n}$ be closed.

Theorem 4 [2] [5, Theorem 9]. If the set A is a connected set totally preordered by closed relation R^k, then there exists a continuous utility function $u_k : A \to R$ such that for every $x, y \in A$ there is $xR^k y \Leftrightarrow u_k(x) \geq u_k(y)$.

One of the most important topological concepts is that of continuity of the utility functions $\{u_k\}_{k=1}^n$.

Let also assume that the set A be compact.

Remark 2. For the utility functions $\{u_k\}_{k=1}^n$ and the optimality sets, it is clear to prove that:

- $M = \bigcap_{k=1}^n Arg\,max(u_k, A)$.
- $Arg\,max(\sum_{k=1}^n u_k, A) \subset E$.
- $\bigcup_{k=1}^n Arg\,max(u_k, A) \subset E_s$

Theorem 5. The set E_s is nonempty compact.

Proof. Let have a convergent sequence $\{x_i\}_{i=1}^\infty \subset E_s \subset A$ and $\lim\limits_{i\to\infty} x_i = x_0$. We know that the set A is compact therefore there is $x_0 \in A$. We will see that $x_0 \in E_s$.

Let assume that $x_0 \notin E$. Then, there exists an alternative $y \in A$ such that $u_k(y) > u_k(x_0)$ for all $k \in [1;n]$.

Let choose $x_j \in \{x_i\}_{i=1}^\infty$. We will see that there exists a number $k \in [1;n]$ such that $u_k(y) \le u_k(x_j)$. Let assume that $u_k(y) > u_k(x_j)$ for all $k \in [1;n]$. This contradicts condition $x_j \in E_s$ therefore we have that there exists a number $k \in [1;n]$ such that $u_k(y) \le u_k(x_j)$.

Hence, there exist a number $k \in [1;n]$ and a sequence $\{x_i'\}_{i=1}^\infty \subset \{x_i\}_{i=1}^\infty$ such that $u_k(y) \le u_k(x_i')$. We have that $\lim\limits_{i\to\infty} x_i = x_0$ therefore we obtain $\lim\limits_{i\to\infty} x_i' = x_0$. The function u_k is continuous on A it follows that $\lim\limits_{i\to\infty} u_k(x_i') = u_k(x_0) \ge u_k(y)$. This contradicts condition $u_k(y) > u_k(x_0)$.

In result, we obtain $x_0 \in E_s$ therefore E_s is closed subset of compact set A. Finally, we obtain the set E_s is compact.

From Remark 2 it follows that the set E_s is nonempty. The theorem is proved.

From this theorem it is clear that $clE_w \subset clE \subset E_s$.

Let denote a function $U : A \to R^n$ such that $U(x) = (u_1(x), u_2(x), ..., u_n(x))$ for all $x \in A$. Thus, we obtain a multi-objective optimization problem
Maximize $U(x)$
Subject to $x \in A$.

In [3], it is also written that an alternative $x \in A$ is called an efficient alternative if and only if there does not exist $y \in A$ such that $U(y) \ge U(x)$ and $U(y) \ne U(x)$. If the set M is nonempty, then $|M| \ge 1$ and $|U(M)| = 1$. See also [11] and [13].

Remark 3. From Remark 2 it follows that the set E is nonempty. The set E_w can be empty or nonempty.

Theorem 6. If $x \in A$, then there exists $y \in E$ such that $y \in \bigcap_{k=1}^n R_k(x)$.

Proof. We shall use a hierarchical method in multi-objective optimization problem [14]. The approach is to rank these to in order of importance $u_1, u_2, ..., u_n$. The problem is optimized on A with respect to u_1 only and we obtain $X_1 = Arg\, max(u_1, \bigcap_{k=1}^{n} R_k(x))$, the set X_1 is nonempty and compact. It is then optimized on X_1 with respect to u_2 only and we obtain $X_2 = Arg\, max(u_2, X_1)$, the set X_2 is nonempty and compact. It is then optimized on X_2 with respect to u_3 only and we obtain $X_3 = Arg\, max(u_3, X_2)$, and so on. Finally, we obtain $X_n = Arg\, max(u_n, X_{n-1})$, the set X_n is nonempty and compact. We also have that $X_n \subset X_{n-1} \subset ... \subset X_1 \subset A$. For this method, see also [8] and [9]. It is proved that $X_n \subset E$ therefore if $y \in X_n$, then $y \in E$ and $y \in \bigcap_{k=1}^{n} R_k(x)$ [15]. The theorem is proved.

Theorem 7. If $|E| = 1$, then $M = E$.

Proof. It is known that $M \subset E$.

Conversely, let $x \in E$ and assume that $x \notin M$. From condition $|E| = 1$ it follows that $E = \{x\}$ and the set M is empty. Therefore, the set $\bigcap_{k=1}^{n} Arg\, max(u_k, A)$ is empty. Then, there exists $j \in [1; n)$ such that a set $\bigcap_{k=1}^{j+1} Arg\, max(u_k, A)$ is empty and a set $\bigcap_{k=1}^{j} Arg\, max(u_k, A)$ is nonempty. Let $y \in \bigcap_{k=1}^{j} Arg\, max(u_k, A)$ and $z \in Arg\, max(u_{j+1}, A)$. From Theorem 6 it follows that $x \in \bigcap_{k=1}^{n} R_k(y)$ and $x \in \bigcap_{k=1}^{n} R_k(z)$. As a result, we obtain $u_k(x) = u_k(y)$ for all $k \in [1; j)$ and $u_{j+1}(x) = u_{j+1}(z)$. Thus, we have that $x \in \bigcap_{k=1}^{j+1} Arg\, max(u_k, A)$. This lead to a contradiction therefore there is $E \subset M$.

Finally, we obtain $M = E$. The theorem is proved.

Corollary 1. The following statements are equivalent:

- The maximum set M is empty.
- $M \neq E$.

Proof. From Remark 3 and Theorem 7 it follows the proof of this corollary.

Corollary 2. The following statements are equivalent:

- The maximum set M is nonempty.
- $M = E$.

Proof. From definitions and Theorem 7 it follows the proof of this corollary.

Corollary 3. (a) If $x, y \in M$, then $u_k(x) = u_k(y)$ for all $k \in [1; n]$.

(b) If $x, y \in E$ and $u_i(x) > u_i(y)$ for some $i \in [1; n]$, then M is empty and there exists $k \in [1; n]$ such that $u_k(y) > u_k(x)$.

Proof. (a) On the basic of the definition of the maximal alternative it follows this proof. See also Remark 2.

(b) From (a) it is clear to show that $\{x, y\} \not\subset M$ therefore there is $M \neq E$. Since Corollary 1 implies that the set M is empty. Let assume that $u_k(x) > u_k(y)$ for all $k \in [1; n]$. Thus, we obtain $x \in \bigcap_{j=1}^{n} R_j(y)$ and $x \in \bigcap_{j=1}^{n} I_j(y)$, this lead to a contradiction. As a result, we have that there exists $k \in [1; n]$ such that $u_k(y) > u_k(x)$. The corollary is proved.

Remark 4. Let analyze the sets E and $U(E)$. Here, we have three cases:

- If $|E| = 1$, then $M = E$ and $|M| = 1$.

- If $|E| > 1$ and $|U(E)| = 1$, then $M = E$ and $|M| > 1$.
- If $|E| > 1$ and $|U(E)| > 1$, then $M \neq E$ and $|M| = 0$.

Corollary 4. If M is empty, then $|E| \geq 2$.

Proof. Let assume that $|E| < 2$ therefore there is $|E| = 1$. From Theorem 7 it follows that M is nonempty and this lead to a contradiction. As a result, we obtain $|E| \geq 2$. The corollary is proved.

Remark 5. Let analyze the set $\bigcap_{k=1}^{n} Arg\,max(u_k, A)$. Here, we have two cases:

- If the set $\bigcap_{k=1}^{n} Arg\,max(u_k, A)$ is nonempty, then $M = E = \bigcap_{k=1}^{n} Arg\,max(u_k, A)$ and $|E| \geq 1$.

- If the set $\bigcap_{k=1}^{n} Arg\,max(u_k, A)$ is empty, then M is empty and $|E| \geq 2$.

On the basic of the definitions and theorems we have the following statements:

- $E = \{x \in A : \{U(x)\} = U(A) \bigcap (U(x) + R_{+}^{n})\}$.
- $E = \{x \in A : \{U(x)\} = (U(A) - R_{+}^{n}) \bigcap \{U(x)\}\}$.
- $E = \{x \in A : \{U(x)\} = (U(A) - R_{+}^{n}) \bigcap (U(x) + R_{+}^{n})\}$.
- $U(M) \subset boU(A)$.
- $U(E) \subset boU(A)$.

For $X \subset R^n$ let denote a set $eff(X) = \{u \in X : v \geq u \wedge v \in X \Rightarrow u = v\}$. It is easy to see that:

- $U^{-1}(eff(U(A))) = E$.
- $E = \{x \in A : U(x) \in eff(U(A))\}$

5. For structure of the optimality sets

In this section, let the utility functions $\{u\}_{k=1}^{n}$ be continuous on the convex and compact set $A \subset R^m$, $m \geq 1$.

It is easy to prove that the sets $\{R_k(x)\}_{k=1}^{n}$ are compact subset of A for all $x \in A$.

A function u_k is concave on A if and only if $x, y \in A$ and $t \in [0;1]$, then $u_k(tx + (1-t)y) \geq tu_k(x) + (1-t)u_k(y)$. A function u_k is quasi-concave on A if and only if $x, y \in A$ and $t \in [0;1]$, then $u_k(tx + (1-t)y) \geq min(u_k(x), u_k(y))$. A function u_k is strictly quasi-concave on A if and only if $x, y \in A$, $x \neq y$ and $t \in (0;1)$, then $u_k(tx + (1-t)y) > min(u_k(x), u_k(y))$.

It is clear to prove that:

- If a function is concave, then it is quasi-concave.
- If a function is strictly quasi-concave, then it is quasi-concave.

In this section, let the functions $\{u\}_{k=1}^{n}$ be concave and strictly quasi-concave on A. In this case, there is $E_w = E = E_s$, see Theorem 3.

Let denote a function $f : A \to R$, $f(x) = \sum_{k=1}^{n} u_k(x)$ for all $x \in A$. It is clear to show that the function f is continuous and concave on A.

Remark 6. It is known that $Arg\,max(f, A) \subset E$ [16], see also Remark 2.

Let denote a point-to-set mapping $\rho : A \to A$ such that $\rho(x) = \{ y \in A : y \in \bigcap_{k=1}^{n} R_k(x) \}$ for all $x \in A$. It is easy to show that the set $\rho(x)$ is a nonempty, convex and compact set for all $x \in A$ and there is $x \in \rho(x)$.

Theorem 8. If $x \in A$, then $|Arg\,max(f, \rho(x))| = 1$ and $Arg\,max(f, \rho(x)) \subset E$.

Proof. It is clear to show that $|Arg\,max(f, \rho(x))| \geq 1$.

Let choose $y_1, y_2 \in Arg\,max(f, \rho(x))$, $y_1 \neq y_2$, $t \in [0;1]$ and $z = ty_1 + (1-t)y_2$. It is known that the set $Arg\,max(f, \rho(x))$ is convex, therefore there is $z \in Arg\,max(f, \rho(x))$. Thus, we obtain $f(z) = f(y_1) = f(y_2)$.

For each $k \in [1;n]$ there is $u_k(z) \geq tu_k(y_1) + (1-t)u_k(y_2)$. By using this result we derive that $f(z) \geq tf(y_1) + (1-t)f(y_2) = f(y_1)$. Since $f(z) = f(y_1)$ implies $u_k(z) = tu_k(y_1) + (1-t)u_k(y_2)$ for all $k \in [1;n]$ and for all $t \in [0;1]$.

In result, we have that $u_k(z) = u_k(y_2) + t(u_k(y_1) - u_k(y_2))$ for all $t \in [0;1]$, therefore we find that $u_k(y_1) = u_k(y_2)$ for all $k \in [1;n]$.

Let choose $t \in (0;1)$ and $k \in [1;n]$. It is known that the function u_k is strictly quasi-concave, therefore we obtain $u_k(z) > min(u_k(y_1), u_k(y_2)) = u_k(y_1)$. But $u_k(z) \geq tu_k(y_1) + (1-t)u_k(y_2)$ and by using this result we have that $f(z) > tf(y_1) + (1-t)f(y_2) = f(y_1)$. This lead to a contradiction, therefore we derive $|Arg\,max(f, \rho(x))| = 1$.

Let choose $y \in Arg\,max(f, \rho(x))$ and assume that $y \notin E$. From condition $y \notin E$ it follows that there exists $z \in A$ such that $u_k(z) \geq u_k(y)$ for all $k \in [1;n]$ and $u_j(z) > u_j(y)$ for some $j \in [1;n]$. As a result, we have that $z \in \rho(x)$ and $f(z) > f(y)$. This lead to a contradiction, therefore we derive $y \in E$, see also [16, Theorem 5]. The theorem is proved.

Theorem 9. Let $x \in A$, $x \in E$ if and only if $\{x\} = \rho(x)$.

Proof. Let $x \in E$ and assume that $\{x\} \neq \rho(x)$. From conditions $x \in \rho(x)$ and $\{x\} \neq \rho(x)$ it follows that there exists $y \in \rho(x) \backslash \{x\}$ such that $u_k(y) \geq u_k(x)$ for all $k \in [1;n]$. Let choose $t \in (0;1)$ and $z = tx + (1-t)y$ therefore $z \in \rho(x)$. Since $x \neq y$ implies $u_k(z) > u_k(x)$ for all $k \in [1;n]$, which contradicts condition $x \in E$ therefore we obtain $\{x\} = \rho(x)$.

Conversely, let $\{x\} = \rho(x)$ and assume that $x \notin E$. From condition $x \notin E$ it follows that there exists $y \in A$ such that $u_k(y) \geq u_k(x)$ for all $k \in [1;n]$ and $u_j(y) > u_j(x)$ for some $j \in [1;n]$. Thus, we have that $y \in \rho(x)$ and $x \neq y$, which contradicts condition $\{x\} = \rho(x)$ therefore we obtain $x \in E$. The theorem is proved.

Let consider the point-to-set mapping ρ. It is easy to show that it is compact-valued mapping.

Lemma 1. If $\{ x_k \}_{k=1}^{\infty}, \{ y_k \}_{k=1}^{\infty} \subset A$ are pair of sequences such that $\lim_{k\to\infty} x_k = x_0 \in X$ and $y_k \in \rho(x_k)$ for all $k \in N$, then there exists a convergent subsequence of $\{ y_k \}_{k=1}^{\infty}$ whose limit belongs to $\rho(x_0)$.

Proof. Since $y_k \in \rho(x_k)$ for all $k \in N$ implies $u_i(y_k) \geq u_i(x_k)$ for all $k \in N$ and all $i \in [1;n]$. From $\{ y_k \}_{k=1}^{\infty} \subset A$ it follows that there exists a convergent sequence $\{ y'_k \}_{k=1}^{\infty} \subset \{ y_k \}_{k=1}^{\infty}$ such that $\lim_{k\to\infty} y'_k = y_0 \in X$, $\{ x'_k \}_{k=1}^{\infty} \subset \{ x_k \}_{k=1}^{\infty}$, $\lim_{k\to\infty} x'_k = x_0$ and $y'_k \in \rho(x'_k)$.

Thus, we have that $u_i(y'_k) \geq u_i(x'_k)$ for all $k \in N$ and for all $i \in [1;n]$.

Taking the limit as $k \to \infty$ we obtain $u_i(y_0) \geq u_i(x_0)$ for all $i \in [1;n]$. As a result, there is $y_o \in \rho(x_0)$. The lemma is proved.

Let denote a distance between an alternative $y \in A$ and a set $X \subset A$ by $d_k = inf\{ d(y_0, x) : x \in X \}$.

Lemma 2. If $\{ x_k \}_{k=1}^{\infty} \subset A$ is a convergent sequence to $x_0 \in A$ and $y_0 \in \rho(x_0)$, then there exists a sequence $\{ y_k \}_{k=1}^{\infty} \subset A$ such that $y_k \in \rho(x_k)$ for all $k \in N$ and $\lim_{k\to\infty} y_k = y_0$.

Proof. From condition the set $\rho(x_k)$ is a nonempty, convex and compact set it follows that:

- If $y_0 \in \rho(x_k)$, then $d_k = 0$ and let $y_k = y_0$.
- If $y_0 \notin \rho(x_k)$, then $d_k > 0$ and there exists a unique projection $y_k \in \rho(x_k)$ on y_0 onto $\rho(x_k)$ such that $d_k = d(y_0, y_k)$.

Therefore, we obtain a sequence $\{ y_k \}_{k=1}^{\infty} \subset A$ such that $y_k \in \rho(x_k)$ for all $k \in N$. Thus, since $\lim_{k\to\infty} x_k = x_0$ implies a sequence $\{ d_k \}_{k=1}^{\infty}$ is convergent and $\lim_{k\to\infty} d_k = 0$. Then, we obtain $\lim_{k\to\infty} y_k = y_0$. The lemma is proved.

Theorem 10. The point-to-set mapping ρ is continuos on A.

Proof. From Lemma 1 it follows that the point-to-set mapping ρ is upper semi-continuous of A [6]. From Lemma 2 it follows that the point-to-set mapping ρ is lower semi-continuous of A [6]. Thus, we obtain the point-to-set mapping ρ is continuous of A. The theorem is proved.

Theorem 11 [5, Theorem 6.5]. Let $X \subset R^m$, $Y \subset R^n$, $n,m \geq 1$, and the set Y be compact, $f : Y \to R$ a continuous function and $\rho : X \to Y$ a continuos point-to-set mapping. Then, the function $m : X \to R$ defined by $m(x) = max\{ f(y) : y \in F(x)\}$ is continuous, and the point-to-set mapping $G : X \to Y$ defined by $G(x) = \{ y \in F(x) : f(y) = m(x)\}$ is upper semi-continuous.

Corollary 5. Let A be compact, $f : A \to R$ a continuous function and $\rho : A \to A$ a continuos point-to-set mapping. Then, the function $m : A \to R$ defined by $m(x) = max\{ f(y) : y \in \rho(x)\}$ is continuous, and the point-to-set mapping $G : A \to A$ defined by $G(x) = \{ y \in \rho(x) : f(y) = m(x)\}$ is upper semi-continuous.

Proof. From Theorem 11 it follows the proof of this corollary.

Remark 7. From Theorem 8 it follows that G is function. In is known that upper semi-continuity for point-to-set mapping is continuity for function. Then, the function G is continuous on A.

Now let analyze a continuous function $G : A \to E$ such that $G(x) \in Arg\,max(f, \rho(x))$ for all $x \in A$.

Corollary 6. $G(A) = E$.

Proof. From $E \subset A$ and Theorem 9 it follows that $G(E) = E$. Then, we obtain $G(A) = E$. The corollary is proved.

Theorem 12. The set E is nonempty, path-connected and compact.

Proof. It is known that every continuous image of a nonempty, path-connected and compact set is a nonempty, path-connected and compact set [6]. From Remark 7 and Corollary 6 it follows that the set E is nonempty, path-connected and compact. The theorem is proved.

Remark 8. It is known that path-connectedness implies connectedness [6], therefore the set E is connected. In [6, Example 1.28 and Remark 1.74], there is an example where it is seen that there exists a connected set that is not path-connected.

Remark 9. If the set M is nonempty, then $M = E = \bigcap_{k=1}^{n} Arg\,max(u_k, A)$ and $|E| \geq 1$ (see Remark 2 and Corollary 2). From Theorem 7 it follows that $|M| = |E| = 1$. In this case, the set E contains a unique maximal alternative.

Remark 10. If the set M is empty, then $|E| \geq 2$ (see Corollary 4). From Theorem12 it follows that the set E is infinite and uncountable.

Remark 11. It is easy to see that for each $x \in A$ the following statements are true:

- $G(x) = G(G(x))$.
- An alternative x is fixed point for G if and only if $x \in E$.

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