

WAVE TYPE SOLUTIONS OF REACTION-DIFFUSION EQUATIONS

Galina Panayotova
 University "Prof. Dr. Asen Zlatarov"
 Burgas, BULGARIA
 e-mail: g_panayotova@aby.bg

Abstract : *In this paper we study the problem for existence of wave type solutions of the reaction-diffusion equation $u_t = \epsilon^2 u_{xx} - f_0(u)$ where $x \in \mathbb{R}$ and $f_0(u) = u(u^2 - 1)$ when $\epsilon \rightarrow 0$.*

Introduction: In this paper we investigate the asymptotic behavior as $\epsilon \rightarrow 0$ of solutions of the reaction-diffusion equation

$$(1) \quad u_t = \epsilon^2 u_{xx} - f_0(u), \quad x \in \mathbb{R} \quad \text{and} \quad f_0(u) = u(u^2 - 1),$$

where u is an order-disorder parameter which is related to the microscopic structure of the matter and $\epsilon > 0$ is a small parameter.

Equation (1) is obtained of

$$(2) \quad u_t = \epsilon^2 u_{xx} - f_a(u)$$

where $x \in (-1, 1)$ and $f_a(u) = (u+a)(u^2 - 1)$ with Neumann boundary conditions for $a=0$. [4]

Equation (2) is perhaps the simplest mathematical model for the dynamical phase transition. The parameter a corresponds to the temperature.

In this paper we study the existence of simple waves for the equation (1), who is equivalent to the system

$$(3) \quad \begin{cases} u_x = v \\ u_t = \epsilon^2 u_{xx} - f_0(u). \end{cases}$$

Presentation:

Definition 1. [1] We say that a solution $(u(t,x), v(t,x))$ of the system (3) is constructed by means of Riemann invariant if it is the form

$$(4) \quad u = U(R(t,x)), \quad v = V(R(t,x))$$

where $U(R)$ and $V(R)$ are functions of a single variable and $R(t,x)$ is a suitable function, called Riemann invariant .

Such solutions are called simple waves when the system (3) homogeneous and simple states when (3) is non-homogeneous.

Lemma 1 [2]. The functions u and v can be represented in the form (4) if and only if

$$u_t v_x - u_x v_t = 0.$$

The proof is standard. Note that is the condition of Lemma1 is satisfied and if u is nondegenerated, i.e.if $\tilde{N}u \neq 0$, then u is Riemann invariant .

Theorem 1 (u,v) is a solution of the system (1), constructed by means of Riemann invariant, exactly when (u,v) is a solution of the system

(5)

$$\begin{cases} u_x = v \\ u_t = e^2 u_{xx} - f_0(u). \\ u_t v_x - u_x v_t = 0. \end{cases}$$

This theorem follows directly from Definition1 and Lemma1.

Let $v_x = y$ and $K(y,u) = e^2 u_{xx} - f_0(u)$.

Now (5) can be represented in the form

(6)

$$\begin{cases} u_x = v \\ u_t = K(y,u) \\ v_x = y \\ v_t = Ky/v. \end{cases}$$

The associated with (6) Pfaff system in R^5 has the form

(7)

$$\begin{cases} \mathbf{w}^1(dz) = du - Kdt - ydx \\ \mathbf{w}^2(dz) = dv - (Ky/v)dt - ydx \end{cases}$$

where $dz = (dt,dx,dy,du,dv)$.

Theorem 2. 1) The system (7) has a solution if and only if there exist vector fields $\mathbf{h}_1, \mathbf{h}_2 \in R^5$ such that $\mathbf{w}^i(\mathbf{h}_k) = 0$ ($i,k = 1,2$) and $[\mathbf{h}_1, \mathbf{h}_2]$ belongs to the linear hull of \mathbf{h}_1 and \mathbf{h}_2 .

2) If $F_I (I=1,2,3)$ denote functionally independent solutions of the system $\mathbf{h}_i F = 0$ ($i=1,2$) (whose existence follows from the conditions 1) in view of Frobenius Theorem), then the system of equations $F_I = C_I$ ($i=1,2,3$), where C_I are suitable constants determined by the initial data functions $y = y(t,x)$, $u = u(t,x)$ and $v = v(t,x)$ then namely the pair (u,v) is a solution of system (6).

Theorem 2 is a particular case of the classical theory of Pfaff's systems. So our initial problem reduces to the problem of finding suitable vector fields \mathbf{h}_1 and \mathbf{h}_2 with properties, described in the conditions 1) of Theorem 2. This is a particular case of general Pfaff problem.

Consider system (7) as an algebraic system of equations with respect to coordinates of the field dz . Since its rank equals 2, the set of its solutions form a linear space (depending on the point $z = (t, x, y, u, v)$), called distribution; denote this distributions by $\mathbf{q}(z)$.

Definition 2 [6]. Every involutive subdistribution $\mathbf{q}_i(z)$ of the distribution $\mathbf{q}(z)$ is called resolving distribution for system (7).

Comparing this definition with condition 1) of Theorem 2, we conclude that in order to reach our goals we must find a resolving distribution of dimension 2 for system (7).

The following three linearly independent vector fields

$$(8) \quad \mathbf{x}_1 = (0, 0, 1, 0, 0), \quad \mathbf{x}_2 = (0, 1, 0, v, y), \quad \mathbf{x}_3 = (1, 0, 0, K, Ky/v)$$

satisfy the Pfaff system (7), i.e. $\mathbf{w}^j(\mathbf{x}_k) = 0$ ($i = 1, 2; k = 1, 2, 3$).

The linear hull of \mathbf{x}_j ($j = 1, 2, 3$) determines a three-dimensional distribution $\mathbf{q}(z)$. If we choose a pair of linearly independent fields $\mathbf{h}_1, \mathbf{h}_2 \in \mathbf{q}(z)$, then their linear hull determines a two-dimensional subdistribution $\mathbf{q}_1(z) \hat{=} \mathbf{q}(z)$. Thus if $[\mathbf{h}_1, \mathbf{h}_2] \hat{=} \mathbf{q}_1(z)$, then $\mathbf{q}_1(z)$ is involutive and from the classical Frobenius' Theorem it follows that $\mathbf{q}_1(z)$ is completely integrable subdistribution of $\mathbf{q}(z)$. Therefore the system $\mathbf{h}_i \mathbf{F} = 0$ ($i=1, 2$) possesses three functionally independent solutions \mathbf{F}_k ($k=1, 2, 3$). So our first task is to build a basis consisting of a pair of vector fields $\mathbf{h}_1, \mathbf{h}_2$, generating an involutive two-dimensional subdistribution $\mathbf{q}_1(z)$ of $\mathbf{q}(z)$.

The vector fields $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ defined above form a basis of the distribution $\mathbf{q}(z)$. Then the following theorems give us a way to find a pair of suitable vector fields $\mathbf{h}_1(z), \mathbf{h}_2(z)$.

Theorem 3. There exists exactly one vector field \mathbf{h}_1 , satisfying the system

$$(9) \quad \mathbf{w}^j(dz) = 0; \quad \mathfrak{f}\mathbf{w}^j(\mathbf{x}_j dz) = 0 \quad (i = 1, 2; j = 1, 2, 3)$$

and exactly one vector field \mathbf{h}_2 , satisfying the system

$$(10) \quad \mathbf{w}^j(dz) = 0; \quad \mathfrak{f}\mathbf{w}^2(\mathbf{x}_j dz) = 0 \quad (i = 1, 2; j = 1, 2, 3)$$

If the system (7) has 2-dimensional resolving distributions $\mathbf{q}_1(z)$, then $\mathbf{h}_1 \hat{=} \mathbf{q}_1(z)$ and $\mathbf{h}_2 \hat{=} \mathbf{q}_1(z)$. For a proof of Theorem 3 see [3], [5] and [7].

Theorem 4. If the system (7) has a two-dimensional resolving distribution $\mathbf{q}_1(z)$, then the vector field

$$\begin{aligned} \mathbf{h}_1 &= (-r/(vK_y - Kr), K_y/(vK_y - Kr), 0, 1, y/v) \\ \mathbf{h}_2 &= (v, -K, s, 0, 0), \end{aligned}$$

where $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are defined by (8), belong to $\mathbf{q}_l(z)$ and $s = ry(vK_y - Kr)/(ryK_y - vK_y + Kr)$.

Proof. We have $[\mathbf{x}_1, \mathbf{x}_2] = \mathbb{F}_y \mathbf{x}_2$; $[\mathbf{x}_1, \mathbf{x}_3] = -\mathbb{F}_y \mathbf{x}_3$; $[\mathbf{x}_2, \mathbf{x}_3] = \mathbb{F}_{\mathbf{x}_2 \mathbf{x}_3} - \mathbb{F}_{\mathbf{x}_3 \mathbf{x}_2} = (0, 0, 0, yK_u, yK_u - y^2K/v^2) = (0, 0, 0, r, yr/v)$, where $r = vK_u - yK/v$.

Hence using the above results, Lemma 1 and the identity $\mathbb{F}[\mathbf{w}(X, Y) = X\mathbf{w}(Y) - Y\mathbf{w}(X) - \mathbf{w}[X, Y]$, we obtain

$\mathbb{F}\mathbf{w}'(\mathbf{x}_1, \mathbf{x}_2) = -\mathbb{F}\mathbf{w}'([\mathbf{x}_1, \mathbf{x}_2]) = 0$; $\mathbb{F}\mathbf{w}'(\mathbf{x}_1, \mathbf{x}_3) = -\mathbb{F}\mathbf{w}'([\mathbf{x}_1, \mathbf{x}_3]) = -K_y$; $\mathbb{F}\mathbf{w}'(\mathbf{x}_2, \mathbf{x}_3) = -\mathbb{F}\mathbf{w}'([\mathbf{x}_2, \mathbf{x}_3]) = -r$; $\mathbb{F}\mathbf{w}^2(\mathbf{x}_1, \mathbf{x}_2) = -\mathbb{F}\mathbf{w}^2([\mathbf{x}_1, \mathbf{x}_2]) = -I$; $\mathbb{F}\mathbf{w}^2(\mathbf{x}_1, \mathbf{x}_3) = -\mathbb{F}\mathbf{w}^2([\mathbf{x}_1, \mathbf{x}_3]) = (yK_y + K)/v$; $\mathbb{F}\mathbf{w}^2(\mathbf{x}_2, \mathbf{x}_3) = -\mathbb{F}\mathbf{w}^2([\mathbf{x}_2, \mathbf{x}_3]) = -yr/v$. Therefore, solving (9) and (10), we get: $\mathbf{h}_{10} = \mathbf{x}_2 - r/K_y \mathbf{x}_3$; $\mathbf{h}_{20} = yr/v \mathbf{x}_1 - (yK_y + K)/v \mathbf{x}_2 + \mathbf{x}_3$.

According to Theorem 3, η_{10} and η_{20} belong to $\mathbf{q}_l(z)$.

We define $\mathbf{h}_1 = ((vK_y - Kr)/K_y)\mathbf{h}_{10}$; $\mathbf{h}_2 = \mathbf{h}_1 - yK_y\mathbf{h}_{20}$, wish can be written in coordinate basis as: $\mathbf{h}_1 = (-r/(vK_y - Kr), K_y/(vK_y - Kr), 0, 1, y/v)$,

$\mathbf{h}_2 = (v, -K, s, 0, 0)$.

The main result

If $[\mathbf{h}_1, \mathbf{h}_2] \hat{\mathbf{I}} \mathbf{q}_l(z)$, then $\mathbf{q}_l(z)$ is involutive and from Frobenius' theorem it follows that $\mathbf{q}_l(z)$ is a completely integrable subdistributions of $\mathbf{q}(z)$.

Theorem 5. The subdistributions $\mathbf{q}_i(z) \hat{\mathbf{I}} \mathbf{q}(z)$ defined as a linear hull of the vectorial fields $\mathbf{h}_i(x)$ ($i=1,2$) is involutive if and only if $r=0$.

Proof. The subdistributions $\mathbf{q}_i(x)$ spanned by the pair of vectorial fields \mathbf{h}_1 and \mathbf{h}_2 is involutive if and only if there exists a linear dependence between $\mathbf{h}_1, \mathbf{h}_2$ and $[\mathbf{h}_1, \mathbf{h}_2]$ i.e. if the rank of the matrix $M^o(\mathbf{h}_1, \mathbf{h}_2, [\mathbf{h}_1, \mathbf{h}_2])$ equals 2. Indeed $[\mathbf{h}_1, \mathbf{h}_2] = (y/v - s[-r/(vK_y - Kr)]_y, -Ku - s[-K_y/(vK_y - Kr)]_y, s_u + us_v/v, 0, s/v)$. Therefore the rank of the M is 2 if and only if $s=0$, i.e. $r=0$.

We obtain for

$$\begin{aligned} \mathbf{h}_1 &= (0, 1/v, 0, 1, y/v), \\ \mathbf{h}_2 &= (v, -K, 0, 0, 0) \end{aligned}$$

Having in mind the classical Frobenius theorem we conclude that equation (1) have wave type solutions, when $r=0$. The Frobenius system is (11)

$$\begin{aligned} \mathbf{h}_1 \mathbf{F} &= 1/v \mathbf{F}_t + \mathbf{F}_u + y/v \mathbf{F}_v = 0 \\ \mathbf{h}_2 \mathbf{F} &= v \mathbf{F}_t - K \mathbf{F}_x = 0 \end{aligned}$$

References

- [1]. Grundland A. "Riemann invariants. In: Wave phenomena: Modern theory and applications", North-Holand Math. Studies 97, Elsevier, New York, 1984.
- [2]. Jeffrey A. "Equations of evolution and waves" In: :Wave phenomena modern theory and applications", North-Holand Math. Studies 97, Elsevier, New York, 1984.

- [3]. Kolev D. "Simple states of non-linear electric field model in \mathbb{R}^2 . *Mathematika Balkanika*, 12 (1998), 315-320.
- [4]. Fusko G. Hale J. Xun J "Traveling waves as limits of solutions on bounded domains " *SIAM J.Math.Anal* Volume 27, N6, 1996,1544-1558.
- [5]. Panayotova G. "Simple waves for certain hyperbolic systems of PDEs" *Mt. Journal of Diff. Equations (Appl. Math)* 2001 (in print).
- [6]. Tabov J. "Simple waves and simple states in \mathbb{R}^2 " *Jornal of Math. Analysis and Applications*", 214(1997), 613-632.
- [7]. Tabov J. and Kolev D. "Solvability of quasi-linear systems in \mathbb{R}^2 .with time-dependent coefficients". *Compt. rend. de l'Acad. bulg. des sci.*,49(1996), 9-10, 35-38.

Technical College - Bourgas,
All rights reserved, © March, 2000