The Optimization Technique in Exchange Mathematical Model with Quasi-concave Utility Functions

Zdravko Dimitrov Slavov
Department of Mathematics, Varna Free University
k."Chaika", Varna 9007, Bulgaria
e-mail: blhrdezd@revolta.com

Abstract: In the presented paper, we study the concept of the Pareto optimality allocations in mathematical model of finite pure exchange economy with perfectly divisible goods and quasi-concave utility functions of the economic agents. We consider three cases of Pareto criterion – weak optimality, strong optimality and full optimality. The examining theorems are based only on the utility of the exchange goods and the quasi-concave utility functions of the economic agents.

Keywords: weak optimality, strong optimality; full optimality; quasi-concave function.

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1. Introduction

The study of pure competition has a central part of pure exchange economy. The problems of competition equilibrium and optimality are basic in finite pure exchange economy with perfectly divisible goods. A main characteristic of the competition equilibrium is given with the Pareto optimality criterion. To define the Pareto optimality criterion: if in the process of allocation of goods between the agents (individuals) in a finite exchange economy the welfare of one single agent increases, without decreasing the welfare of all the other agents, then the welfare of the economy as a whole increases. Following we obtain the definition of Pareto optimality allocation: the allocation of goods is Patero optimality if and only if it is not possible for the welfare of a certain agent to be improved without involving the worsening of the welfare of another agent. It is proved that the competition equilibrium allocations of goods are Pareto optimality. We will examine a number of characteristics of the Pareto optimality allocations not using the fact of competition equilibrium. This is an important issue because Pareto optimality allocations do not use the price system and budgetary limitations of the agents.

Quasi-concavity plays a central role in mathematical model of finite pure exchange economy and optimization theory. Therefore, the research on quasi-concavity is one of the most important aspects of the optimization technique in finite pure exchange economy with perfectly divisible goods and quasi-concave utility functions of the agents.

2. Notations and assumptions

We consider a mathematical model of finite pure exchange economy e(A,G,D,U). Let A be a set of agents, let G be a set of goods and let each agent $a_i \in A$ has consumption set $X^i \subset \Re^m_+$ and endowment $w^i(w^i_1, w^i_2, ..., w^i_m) \in X^i$, where $w^i_j \geq 0$ shows the quantity of good $g_j \in G$ property of agent $a_i \in A$. Let for each good $g_j \in G$ we have $v_j = \sum_{i=1}^n w^i_j > 0$, $v(v_1, v_2, ..., v_m) \in \Re^m_{++}$ and let for each agent $a_i \in A$ we have a following assumption:

(A1.1) The set X^i is convex and compact.

Here we denote

$\Re_{+}^{m} = \{x : x(x_1, x_2, ..., x_m) \in \Re^m \ \& \ x_j \geq 0 \forall j \in [1; m]\}$

$\Re_{++}^{m} = \{x : x(x_1, x_2, ..., x_m) \in \Re^m \ \& \ x_j > 0 \forall j \in [1; m]\}$

The set $D = \{x : x(x^1, x^2, ..., x^n) \in X^1 \times X^2 \times ... \times X^n \ \& \sum_{i=1}^{n} x^i = v\}$ we will call the set of individually rational allocations. It is proved that the set D is nonempty, convex and compact. Here, we use the assumption A1.1.

Let we have $|A| \geq 2$ and $|G| \geq 2$, and let the sets $\{X^i\}_{i=1}^{n} \subset \Re_{+}^{m}$ are such that $|D| \geq 2$.

Let $U = \{u_i\}_{i=1}^{n}$ is a profile of functions on D and each agent $a_i \in A$ has the utility function $u_i : D \to \Re$ with following assumptions:

(A2.1) If $x, y \in D$ and $x^i = y^i$, then $u_i(x) = u_i(y)$;

(A2.2) The utility function u_i is continuous on D.

If $x \in D$, then the set $R_i(x) = \{y : y \in D \ \& \ u_i(y) \geq u_i(x)\}$ we will call the set of weakly preference of the agent $a_i \in A$. It is proved that the sets $R_i(x)$ and $\bigcap_{i=1}^{n} R_i(x)$ are nonempty and compact, $x \in R_i(x)$ and $x \in \bigcap_{i=1}^{n} R_i(x)$. Here, we use the assumptions A2.1 and A2.2.

If $x \in D$, then the set $I_i(x) = \{y : y \in D \ \& \ u_i(y) = u_i(x)\}$ we will call the set of indifference of the agent $a_i \in A$. It is proved that the sets $I_i(x)$ and $\bigcap_{i=1}^{n} I_i(x)$ are nonempty and compact, $x \in I_i(x)$ and $x \in \bigcap_{i=1}^{n} I_i(x)$. Here, we use the assumptions A2.1 and A2.2.

If $x \in D$, then the set $P_i(x) = \{y : y \in D \ \& \ u_i(y) > u_i(x)\}$ we will call the set of strongly preference of the agent $a_i \in A$. The sets $P_i(x)$ and $\bigcap_{i=1}^{n} P_i(x)$ can be empty, $x \notin P_i(x)$ and $x \notin \bigcap_{i=1}^{n} P_i(x)$.

It is clear to show that if $x \in D$, then $I_i(x) \subset R_i(x)$ and $P_i(x) \subset R_i(x)$ for all $a_i \in A$, $\bigcap_{i=1}^{n} I_i(x) \subset \bigcap_{i=1}^{n} R_i(x)$ and $\bigcap_{i=1}^{n} P_i(x) \subset \bigcap_{i=1}^{n} R_i(x)$. We have also $I_i(x) \bigcap P_i(x)$ is empty and $R_i(x) = I_i(x) \bigcup P_i(x)$.

We will make some addition assumptions, which will hold in some of the next theorems.

The utility function u_i of $a_i \in A$ we will call quasi-concave if and only if $x, y \in D$ and $t \in [0;1]$, then $u_i(tx + (1-t)y) \geq \min(u_i(x), u_i(y))$.

Assumption 3.1. For each agent $a_i \in A$ the utility function u_i is quasi-concave.

The utility function u_i of $a_i \in A$ we will call strictly quasi-concave if and only if $x, y \in D$, $x^i \neq y^i$ and $t \in (0;1)$, then $u_i(tx + (1-t)y) > \min(u_i(x), u_i(y))$.

Assumption 3.2. For each agent $a_i \in A$ the utility function u_i is strictly quasi-concave.

It is clear to show that:

(i) if Assumption 3.2 holds, then Assumption 3.1 holds;

(ii) if Assumption 3.1 or 3.2 holds, then the sets $R_i(x)$, $P_i(x)$, $\bigcap_{i=1}^{n} R_i(x)$ and $\bigcap_{i=1}^{n} P_i(x)$ are convex.

3. Main characteristics of the optimality allocations

Definition 1. An allocation $y \in D$ weakly dominates an allocation $x \in D$ if and only if $u_i(y) \geq u_i(x)$ for all $a_i \in A$ and $x \neq y$. We will call the allocation $x \in D$ is weak optimality if and only if there does not exist $y \in D$ such that y weakly dominates x. The set of the weak optimality allocations of D we will be denoted by P_w.

Remark 1. Any authors call the elements of set P_w strong optimality.

Lemma 1. Let $x \in D$, the following statements are equivalent:

(S1.1) $x \in P_w$;

(S1.2) $\{ y : y \in D,\ \forall a_i \in A\ \ u_i(y) \geq u_i(x)$ and $y \neq x \}$ is empty;

(S1.3) $\{ y : y \in D,\ \forall a_i \in A\ \ u_i(y) \geq u_i(x)$ and $\exists a_k \in A\ \ y^k \neq x^k \}$ is empty;

(S1.4) $\{x\} = \bigcap_{i=1}^{n} R_i(x)$.

Proof. From Definition 1 it follows the proof of Lemma 1.

Definition 2. An allocation $y \in D$ strongly dominates an allocation $x \in D$ if and only if $u_i(y) \geq u_i(x)$ for all $a_i \in A$ and $u_k(y) \neq u_k(x)$ for some $a_k \in A$. We will call the allocation $x \in D$ is strong optimality if and only if there does not exist $y \in D$ such that y strongly dominates x. The set of the strong optimality allocations of D we will be denoted by P_s.

Remark 2. Any authors call the elements of set P_s weak optimality.

Lemma 2. Let $x \in D$, the following statements are equivalent:

(S2.1) $x \in P_s$;

(S2.2) $\{ y : y \in D,\ \forall a_i \in A\ \ u_i(y) \geq u_i(x)$ and $\exists a_k \in A\ \ u_k(y) \neq u_k(x) \}$ is empty;

(S2.3) $\{ y : y \in D,\ \forall a_i \in A\ \ u_i(y) \geq u_i(x)$ and $\exists a_k \in A\ \ u_k(y) > u_k(x) \}$ is empty.

Proof. From Definition 2 it follows the proof of Lemma 2.

Theorem 1. The set P_s is nonempty.

Proof. Let function $f : D \to \Re$ defined by $f(x) = \sum_{i=1}^{n} u_i(x)$ for $x \in D$. From continuity of the functions $\{u_i\}_{i=1}^{n}$ on D it follows the function f is continuous on D. The set D is compact, therefore there exists $x \in D$ such that $f(x) = \sup\{f(z) : z \in D\}$.

Let us assume that $x \notin P_s$, therefore there exists $y \in D$ such that $u_i(y) \geq u_i(x)$ for all $a_i \in A$ and $u_k(y) > u_k(x)$ for some $a_k \in A$, therefore $f(y) > f(x)$. This contradicts with $f(x) = \sup\{f(z) : z \in D\}$, therefore $x \in P_s$ and the set P_s is nonempty. The theorem is proved.

We use assumption A2.2 and statement S2.3 in the proof of Theorem 1.

Definition 3. An allocation $y \in D$ fully dominates an allocation $x \in D$ if and only if $u_i(y) > u_i(x)$ for all $a_i \in A$. We will call the allocation $x \in D$ is full optimality if and only if there does not exist $y \in D$ such that y fully dominates x. The set of the full optimality allocations of D we will be denoted by P_f.

Remark 3. Any authors call the elements of set P_f weak optimality or strong optimality.

Lemma 3. Let $y \in D$, the following statements are equivalent:

(S3.1) $x \in P_f$;

(S3.2) $\{ y : y \in D,\ \forall a_i \in A\ \ u_i(y) > u_i(x) \}$ is empty;

(S3.3) $\bigcap_{i=1}^{n} P_i(x)$ is empty.

Proof. From Definition 3 it follows the proof of Lemma 3.

From Definitions 1, 2 and 3 it is clear to show that the Pareto optimality is not related to the price system of goods and it is defined only by the utility functions of the economic agents.

Theorem 3. The set P_f is nonempty and compact.

Proof. Let function $f : D \to \Re$ defined by $f(x) = \sum_{i=1}^{n} u_i(x)$ for $x \in D$. From continuity of the functions $\{u_i\}_{i=1}^{n}$ on D it follows the function f is continuous on D. The set D is compact, therefore there exists $x \in D$ such that $f(x) = \sup\{f(z) : z \in D\}$.

Let us assume that $x \notin P_f$, therefore there exists $y \in D$ such that $u_i(y) > u_i(x)$ for all $a_i \in A$, therefore $f(y) > f(x)$. This contradicts with condition $f(x) = \sup\{f(z) : z \in D\}$, therefore $x \in P_f$ and the set P_f is nonempty.

Let we have a convergent sequence $\{x_i\}_{i=1}^{\infty} \subset P_f \subset D$ and $\lim_{i \to \infty} x_i = x_0$. The set D is compact therefore $x_0 \in D$. We will prove that $x_0 \in P_f$.

Let us assume that $x_0 \notin P_f$ it follows there exists $y \in D$ such that $u_i(y) > u_i(x_0)$ for all $a_i \in A$.

Let $x_k \in \{x_i\}_{i=1}^{\infty}$, we will show that there exists $a_i \in A$ such that $u_i(y) \le u_i(x_k)$. Let us assume $u_i(y) > u_i(x_k)$ for all $a_i \in A$. This contradicts with $x_k \in P_f$, therefore there exists $a_i \in A$ such that $u_i(y) \le u_i(x_k)$.

The set A is finite it follows that there exist $a_i \in A$ and subsequence $\{x'_i\}_{i=1}^{\infty} \subset \{x_i\}_{i=1}^{\infty}$ such that $u_i(y) \le u_i(x'_k)$. We have $\lim_{i\to\infty} x_i = x_0$ therefore we obtain $\lim_{i\to\infty} x'_i = x_0$. The function u_i is continuous it follows $\lim_{i\to\infty} u_i(x'_i) = u_i(x_0) \ge u_i(y)$. This contradicts with condition $u_i(y) > u_i(x_0)$. In result we obtain $x_0 \in P_f$ therefore the set P_f is closed subset of compact set D. Finally, we obtain the set P_f is compact. The theorem is proved.

Theorem 3. $P_w \subset P_s \subset P_f$.

Proof. Let $x \in P_w$ and let us assume that $x \notin P_s$, therefore there exists $y \in D$ such that $u_i(y) \ge u_i(x)$ for all $a_i \in A$ and $u_k(y) > u_k(x)$ for some $a_k \in A$. In result we obtain $y \in \bigcap_{i=1}^{n} R_i(x) = \{x\}$, therefore we have $x = y$. This contradicts with condition $u_k(y) > u_k(x)$, therefore $P_w \subset P_s$.

Let $x \in P_s$ and let us assume that $x \notin P_f$, therefore there exists $y \in D$ such that $u_i(y) > u_i(x)$ for all $a_i \in A$. In result we obtain $u_i(y) \ge u_i(x)$ for all $a_i \in A$ and $u_k(y) > u_k(x)$ for some $a_k \in A$. This contradicts with $x \in P_s$, therefore we have $P_s \subset P_f$. The theorem is proved.

We use statements S1.4, S2.3 and S3.2 in the proof of Theorem 3.

Theorem 4. If $a_i \in A$ and H is nonempty and compact subset of D, then

$$\hat{H} = \{\, x : x \in H,\ u_i(x) \ge u_i(y) \text{ for all } y \in H \,\}$$

is nonempty and compact subset of H.

(a) If Assumption 3.1 holds, then $\hat{H}$ convex;

(b) If Assumption 3.2 holds and $x, y \in \hat{H}$, then $x^i = y^i$.

Proof. The set H is compact and the function u_i is continuous, therefore there exists $x \in H$ such that $u_i(x) = \sup\{u_i(y) : y \in H\}$. In result we have that the set $\hat{H}$ is nonempty. From continuity of the function u_i it follows the set $\hat{H}$ is a closed subset of the compact set H, therefore $\hat{H}$ is a compact subset of H.

a) Let $x, y \in \hat{H}$, $t \in [0;1]$ and $z = tx + (1-t)y$. From $x, y \in \hat{H}$ it follows $u_i(x) = u_i(y) \ge u_i(z)$. From Assumption 3.1 we have $u_i(z) \ge \min(u_i(x), u_i(y)) = u_i(x)$. In result there is $u_i(z) = u_i(x) = \sup\{u_i(y) : y \in H\}$, therefore $z \in \hat{H}$. Finally, we obtain the set $\hat{H}$ is convex.

b) Let $x, y \in \hat{H}$, $t \in (0;1)$ and $z = tx + (1-t)y$. From $x, y \in \hat{H}$ it follows $u_i(x) = u_i(y) \ge u_i(z)$. Let us assume $x^i \ne y^i$, from Assumption 3.2 we have $u_i(z) > \min(u_i(x), u_i(y)) = u_i(x)$, which contradicts the condition $u_i(x) = u_i(y) \ge u_i(z)$. In result we obtain $x^i = y^i$. The theorem is proved.

We use assumptions A2.1 and A2.2 in the proof of Theorem 4.

Theorem 5. Let Assumption 3.2 holds, if $x \in D$, then there exists $y \in P_w$ such that $u_i(y) \ge u_i(x)$ for all $a_i \in A$.

Proof. Let H=$\bigcap_{i=1}^{n} R_i(x)$, we have that the set H is nonempty and compact. Let

$$\hat{H}_1 = \{\, x : x \in H,\ u_1(x) \ge u_1(y) \text{ for all } y \in H \,\}.$$

From Theorem 4 it follows the set $\hat{H}_1$ is nonempty and compact. Let

$$\hat{H}_2 = \{ x : x \in \hat{H}_1,\ u_2(x) \geq u_2(y) \text{ for all } y \in \hat{H}_1 \}.$$

From Theorem 4 it follows the set $\hat{H}_2$ is nonempty and compact. Let

$$\hat{H}_3 = \{ x : x \in \hat{H}_2,\ u_3(x) \geq u_3(y) \text{ for all } y \in \hat{H}_2 \}.$$

From Theorem 4 it follows the set $\hat{H}_3$ is nonempty and compact, and so on let

$$\hat{H}_n = \{ x : x \in \hat{H}_{n-1},\ u_n(x) \geq u_n(y) \text{ for all } y \in \hat{H}_{n-1} \}.$$

From Theorem 4 it follows the set $\hat{H}_n$ is nonempty and compact.

In result we obtain

$$\hat{H}_n \subset \hat{H}_{n-1} \subset \ldots \subset \hat{H}_2 \subset \hat{H}_1 \text{ and } \hat{H}_n = \bigcap_{i=1}^{n} \hat{H}_i .$$

Let $y, z \in \hat{H}_n$, therefore we have $u_i(y) = u_i(z) \geq u_i(x)$ for all $a_i \in A$. From Theorem 4 we obtain $y^i = z^i$ for all $a_i \in A$ therefore $x = y$. We obtain $\{y\} = \bigcap_{i=1}^{n} R_i(y)$. Finally, there is $y \in P_w$. The theorem is proved.

We use assumption A2.2 and statement S1.4 in the proof of Theorem 5.

Corollary 1. If Assumption 3.2 holds, then the set P_w is nonempty.

Proof. There is $w \in D$ therefore from Theorems 4 and 5 it follows the proof of Corollary 1.

Corollary 2. (a) Let Assumption 3.2 holds, if $x \in D$, then there exists $y \in P_s$ such that $u_i(y) \geq u_i(x)$ for all $a_i \in A$.

(b) Let Assumption 3.2 holds, if $x \in D$, then there exists $y \in P_f$ such that $u_i(y) \geq u_i(x)$ for all $a_i \in A$.

Proof. From Theorems 3 and 5 it follows the proof of Corollary 2.

Theorem 6. If Assumption 3.2 holds, then $P_w = P_s$.

Proof. Let $x \in P_s$ and let us assume that $x \notin P_w$, therefore there exists $y \in D$ such that $u_i(z) \geq u_i(x)$ for all $a_i \in A$ and $x \neq y$.

Let $t \in (0;1)$ and $z = tx + (1-t)y$. The set D is convex therefore $z \in D$. We have two cases:

First, if $a_i \in A$ and $x^i = y^i$, then $z^i = x^i$, therefore $u_i(z) = u_i(x)$;

Second, if $a_i \in A$ and $x^i \neq y^i$, then $u_i(z) > \min(u_i(x), u_i(y)) = u_i(x)$.

In result we obtain $u_i(z) \geq u_i(x)$ for all $a_i \in A$.

From $x \neq y$ it follows there exists $a_k \in A$ such that $x^k \neq y^k$, therefore $u_k(z) > u_k(x)$. In result we have $u_i(z) \geq u_i(x)$ for all $a_i \in A$ and $u_k(z) > u_k(x)$, which contradicts the condition $x \in P_s$ therefore $x = y$. We obtain $\{x\} = \bigcap_{i=1}^{n} R_i(x)$ therefore $x \in P_w$. It is follows $P_s \subset P_w$.

Finally, from Theorem 6 we obtain $P_w = P_s$. The theorem is proved.

We use assumption A2.1 and statements S1.4 and S2.3 in the proof of Theorem 6.

Theorem 7. If Assumption 3.2 holds and $n = 2$, then $P_s = P_f$.

Proof. Let $x \in P_f$ and let us assume that $x \notin P_s$, therefore there exists $y \in D$ such that $u_i(y) \geq u_i(x)$ for all $a_i \in A$ and $u_k(y) > u_k(x)$ for some $a_k \in A$.

We will show that $y^k \neq x^k$. If $y^k = x^k$, then $u_k(y) = u_k(x)$, which contradicts the condition $u_k(y) > u_k(x)$. Therefore we obtain $y^k \neq x^k$.

There is $\sum_{i=1}^{2} y^i = \sum_{i=1}^{2} x^i$. From $y^k \neq x^k$ it follows $y^l \neq x^l$ for $l \neq k$.

Let $t \in (0;1)$ and $z = tx + (1-t)y$. The set D is convex therefore $z \in D$. We obtain:

First, from $y^k \neq x^k$ we have $u_k(z) > \min(u_k(y), u_k(x)) = u_k(x)$.

Second, from $y^l \neq x^l$ we have $u_l(z) > \min(u_l(y), u_l(x)) = u_l(x)$.

In result we obtain $u_i(z) > u_i(x)$ for all $a_i \in A$, which contradicts the condition $x \in P_f$. It is follows $P_f \subset P_s$.

Finally, from Theorem 6 we obtain $P_s = P_f$. The theorem is proved.

We use assumption A2.1 and statements S2.3 and S3.2 in the proof of Theorem 7.

Corollary 3. If Assumption 3.2 holds and $n = 2$, then $P_w = P_s = P_f$.

Proof. From Theorems 6 and 7 it follows the proof of Corollary 3.

Corollary 4. If Assumption 3.2 holds and $n = 2$, then the sets P_w and P_s are compact.

Proof. From Theorem 2 and Corollary 3 it follows the proof of Corollary 4.

4. Examples and remarks

We will study some special examples. Here, we will analyze the basic characteristics of optimality sets P_w, P_s and P_f.

Example 1. Let $\{u_i\}_{i=1}^{n} \subset \Re$ and $u_i(x) = u_i$ for all $x \in D$ and for all $a_i \in A$.

First, we will consider the weak optimality allocations. If $a_i \in A$ and $x \in D$, then $u_i(y) \geq u_i(x)$ for all $y \in D$. We obtain the set

$$\{\, y : y \in D,\ \forall a_i \in A \ \ u_i(y) \geq u_i(x) \text{ and } y \neq x \,\}$$

is nonempty. It follows the set P_w is empty.

Second, we will consider the strong optimality allocations. If $a_i \in A$ and $x \in D$, then $u_i(y) \geq u_i(x)$ for all $y \in D$ and there does not exist $y \in D$ such that $u_k(y) \neq u_k(x)$ for some $a_k \in A$. We obtain the set

$$\{\, y : y \in D,\ \forall a_i \in A \ \ u_i(y) \geq u_i(x) \text{ and } \exists a_k \in A \ \ u_k(y) \neq u_k(x) \,\}$$

is empty for all $x \in D$. It follows $P_s = D$.

Third, we will consider the full optimality allocations. If $a_i \in A$ and $x \in D$, then there does not exist $y \in D$ such that $u_i(y) > u_i(x)$ for all $a_i \in A$. We obtain the set

$$\{\, y : y \in D,\ \forall a_i \in A \ \ u_i(y) > u_i(x) \,\}$$

is empty for all $x \in D$. It follows $P_f = D$.

Incidentally, from Theorem 3 we have $P_s \subset P_f \subset D$ and from $P_s = D$ it follows $P_f = D$.

In this example we obtain:

(1) The Assumption 3.1 holds;

(2) The Assumption 3.2 does not hold;

(3) The set P_w is empty, $P_w \neq P_s$ and $P_s = P_f$.

Remark 4. From Example 1 it follows the set P_w can be empty.

Example 2. Let $X^i = \{x : x(x_1, x_2, ..., x_m) \in \Re^m \ \& \ 0 \leq x_j \leq v_j \forall j \in [1;m]\}$ for $i \in [1;n]$, $\{u_i\}_{i=1}^{n-1} \subset \Re$ and $u_i(x) = u_i$ for all $x \in D$ and for all $a_i \in A \setminus \{a_n\}$, and $u_n(x) = \sum_{j=1}^{m} x_j^n$ for all $x \in D$.

Let $z \in D$ shut that $z^i = 0$ for all $a_i \in A \setminus \{a_n\}$ and $z^n = v$.

First, we will consider the weak optimality allocations. We obtain the set

$$\{\, y : y \in D,\ \forall a_i \in A \ \ u_i(y) \geq u_i(z) \text{ and } y \neq z \,\}$$

is empty, therefore $z \in P_w$. If $x \in D \setminus \{z\}$, then

$$z \in \{\, y : y \in D,\ \forall a_i \in A \ \ u_i(y) \geq u_i(x) \text{ and } y \neq x \,\}.$$

In result we obtain $P_w = \{z\}$ and the set P_w is nonempty.

Second, we will consider the strong optimality allocations. We have $u_i(y) \geq u_i(z)$ for all $a_i \in A$ and $u_n(z) > u_n(y)$ for all $y \in D \setminus \{z\}$. We obtain the set

$$\{\, y : y \in D\,,\ \forall a_i \in A \ \ u_i(y) \ge u_i(z) \text{ and } u_n(y) > u_n(z) \}$$

is empty, therefore $z \in P_s$. If $x \in D \setminus \{z\}$, then

$$z \in \{\, y : y \in D\,,\ \forall a_i \in A \ \ u_i(y) \ge u_i(x) \text{ and } u_n(y) > u_n(x) \,\}.$$

In result we obtain $P_s = \{z\}$.

Third, we will consider the full optimality allocations. We have there does not exist $y \in D$ such that $u_i(y) > u_i(x)$ for all $a_i \in A$. We obtain the set

$$\{\, y : y \in D\,,\ \forall a_i \in A \ \ u_i(y) > u_i(x) \,\}$$

is empty for all $x \in D$. It follows $P_f = D$.

In this example we obtain:

(1) The Assumption 3.1 holds;

(2) The Assumption 3.2 does not hold;

(3) The set P_w is nonempty, $P_w = P_s$ and $P_s \ne P_f$.

Remark 5. From Example 2 it follows the set P_w can be nonempty.

Remark 6. From Examples 1 and 2 we obtain the following statement: if there exists $a_k \in A$ such that $u_k(x) = u_k(y)$ for all $x, y \in D$, then $P_f = D$. We use that there does not exist $y \in D$ such that $u_i(y) > u_i(x)$ for all $a_i \in A$ in the proof of this statement.

Example 3. Let for each $a_i \in A$ we have $u_i(x) = \sum_{j=1}^{m} x_j^i$ for all $x \in D$.

Let $x, y \in D$ are such that $u_i(y) \ge u_i(x)$ for all $a_i \in A$. We have

$$\sum_{i=1}^{n} \sum_{j=1}^{m} w_j^i = \sum_{i=1}^{n} \sum_{j=1}^{m} y_j^i = \sum_{i=1}^{n} \sum_{j=1}^{m} x_j^i .$$

In result we obtain $u_i(y) = u_i(x)$ for all $a_i \in A$.

First, we will consider the weak optimality allocations. We obtain the set

$$\{\, y : y \in D\,,\ \forall a_i \in A \ \ u_i(y) \ge u_i(x) \text{ and } y \ne x \,\}$$

is nonempty for all $x \in D$, i.e. the set P_w is empty.

Second, we will consider the strong optimality allocations. We obtain the set

$$\{\, y : y \in D\,,\ \forall a_i \in A \ \ u_i(y) \ge u_i(x) \text{ and } \exists a_k \in A \ \ u_k(y) > u_k(x) \,\}$$

is empty for all $x \in D$, therefore $P_s = D$.

Third, we will consider the full optimality allocations. We obtain the set

$$\{\, y : y \in D\,,\ \forall a_i \in A \ \ u_i(y) > u_i(x) \,\}$$

is empty for all $x \in D$, therefore $P_f = D$.

Incidentally, from Theorem 2 we have $P_s \subset P_f \subset D$ and from $P_s = D$ it follows $P_f = D$.

In this example we obtain:

(1) The Assumption 3.1 holds;

(2) The Assumption 3.2 does not hold;

(3) The set P_w is empty, $P_w \ne P_s$ and $P_s = P_f$.

Remark 7. In Remark 6 we obtain the statement: if there exists $a_k \in A$ such that $u_k(x) = u_k(y)$ for all $x, y \in D$, then $P_f = D$. From Example 3 we obtain that the converse statement does not hold: if $P_f = D$, then there exist $a_k \in A$ such that $u_k(x) = u_k(y)$ for all $x, y \in D$.

Example 4. Let for each $a_i \in A$ we have $u_i(x) = \sum_{j=1}^{m} \sqrt{x_j^i}$ for all $x \in D$.

Function $y = \sqrt{x}$ for $x \ge 0$ is strictly concave, because $y'' = -\dfrac{1}{4\sqrt{x^3}} < 0$ for all $x > 0$. Therefore it is strictly quasi-concave. In result we obtain that the functions $\{u_i\}_{i=1}^{n}$ are strictly quasi-concave.

From Theorem 6 it follows $P_w = P_s$.

In this example we obtain:

(1) The Assumption 3.1 holds;

(2) The Assumption 3.2 holds;

(3) The set P_w is nonempty and $P_w = P_s$.

Remark 8. There is $P_w = P_s = P_f$ for $n = 2$ in Example 4. See Theorem 6, Theorem 7 and Corollary 3.

Example 5. Let for each $a_i \in A$ we have $u_i(x) = \sum_{j=1}^{m} \ln(1 + x_j^i)$ for all $x \in D$.

Function $y = \ln(1 + x)$ for $x \ge 0$ is strictly concave, because $y'' = -\dfrac{1}{(1+x)^2} < 0$ for all $x \ge 0$. Therefore it is strictly quasi-concave. In result we obtain that the functions $\{u_i\}_{i=1}^{n}$ are strictly quasi-concave.

From Theorem 6 it follows $P_w = P_s$.

In this example we obtain:

(1) The Assumption 3.1 holds;

(2) The Assumption 3.2 holds;

(3) The set P_w is nonempty and $P_w = P_s$.

Remark 9. There is $P_w = P_s = P_f$ for $n = 2$ in Example 5. See Theorem 6, Theorem 7 and Corollary 3.

Example 6. Let $n = m \ge 2$, $X^i = \{x : x(x_1, x_2, ..., x_m) \in \Re^m \;\&\; 0 \le x_i \le v_i\}$ for $i \in [1;n]$, and for each $a_i \in A$ we have $u_i(x) = x_i^i$ for all $x \in D$.

Let $z \in D$ shut that for each $i \in [1;n]$ we have $z_i^i = v_i$ and $z_j^i = 0$ for $j \in [1;n] \setminus \{i\}$.

First, we will consider the weak optimality allocations. We obtain the set

$$\{\, y : y \in D,\ \forall a_i \in A\ \ u_i(y) \ge u_i(z) \text{ and } y \ne z \,\}$$

is empty, therefore $z \in P_w$. If $x \in D \setminus \{z\}$, then

$$z \in \{\, y : y \in D,\ \forall a_i \in A\ \ u_i(y) \ge u_i(x) \text{ and } y \ne x \,\}.$$

In result we obtain $P_w = \{z\}$ and the set P_w is nonempty.

Second, we will consider the strong optimality allocations. Let $y \in D \setminus \{z\}$ such that we have $u_i(y) \ge u_i(z)$ for all $a_i \in A$ and $u_k(z) > u_k(y)$ for some $a_k \in A$. We obtain the set

$$\{\, y : y \in D,\ \forall a_i \in A\ \ u_i(y) \ge u_i(z) \text{ and } \forall a_k \in A\ \ u_k(y) > u_k(z) \,\}$$

is empty, therefore $z \in P_s$. If $x \in D \setminus \{z\}$, then

$$z \in \{\, y : y \in D,\ \forall a_i \in A\ \ u_i(y) \ge u_i(x) \text{ and } \exists a_k \in A\ \ u_k(y) > u_k(x) \,\}.$$

In result we obtain $P_s = \{z\}$.

Third, we will consider the full optimality allocations. Let $y \in D \setminus \{z\}$ such that we have $u_i(y) > u_i(z)$ for all $a_i \in A$. We obtain the set

$$\{\, y : y \in D,\ \forall a_i \in A\ \ u_i(y) > u_i(z) \,\}$$

is empty, therefore $z \in P_f$. If $x \in D \setminus \{z\}$, then

$$z \in \{\, y : y \in D,\ \forall a_i \in A\ \ u_i(y) > u_i(x) \,\}.$$

In result we obtain $P_f = \{z\}$.

In this example we obtain:

(1) The Assumption 3.1 holds;

(2) The Assumption 3.2 does not hold;

(3) The set P_w is nonempty and $P_w = P_s = P_f$.

Remark 10. Let $n = m > 2$ in Example 6. In this case we have $P_w = P_s = P_f$.

Remark 11. For Example 6 it follows $P_s = P_f$ and $P_f \neq D$.

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